Practical Applications of the L1 penalty

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StatLearn, April 2022

Outline

Outline

Before we start

Jobs in ML/AI

Source: Les Décodeuses du numérique

Before we start

Jobs in ML/AI

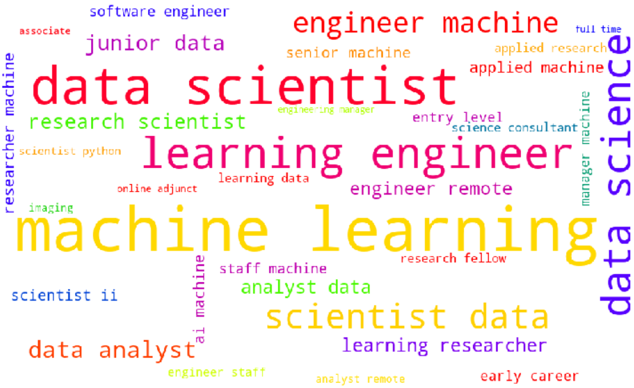
"There is the royal way, getting an AI job in a company and the imperial way, getting an AI job in academia"

Stéphane Canu

Source: Les Décodeuses du numérique

Before we start

Jobs in ML/AI



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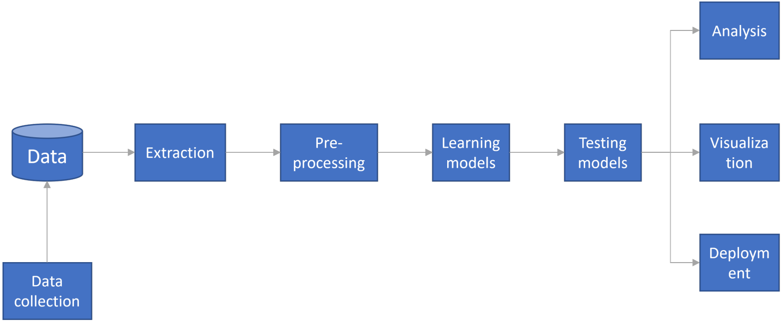
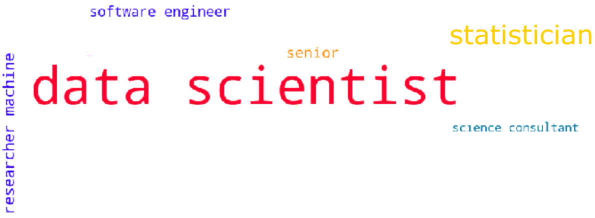


Data engineer ML engineer Data Scientist AI researcher Data analyst

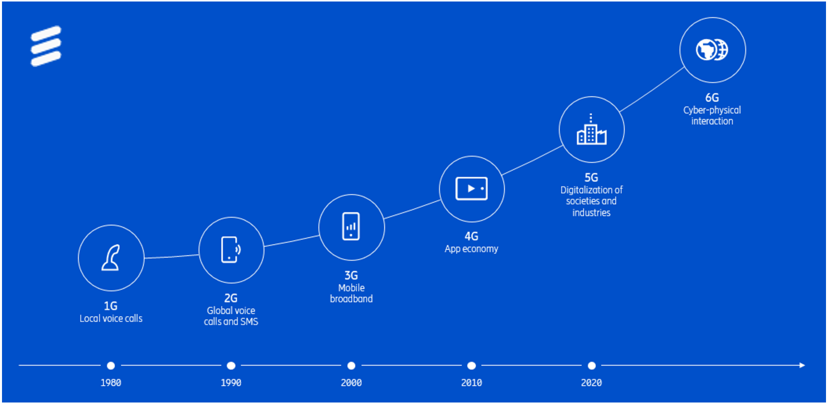
Before we start

My personal experience



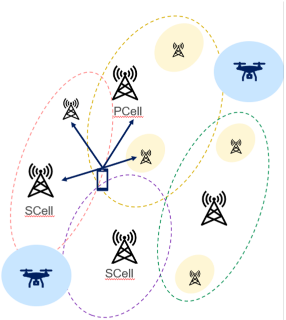
AI in telecom and at Ericsson

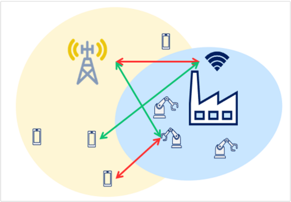
- AI is progressively being integrated into 5G and 6G networks

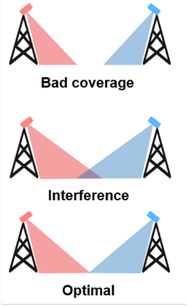


AI in telecom and at Ericsson

- AI is progressively being integrated into 5G and 6G networks
- AI & Systems team started in Paris area (near Saclay) in November
  - Main topics: reinforcement learning, transfer learning, sparse models

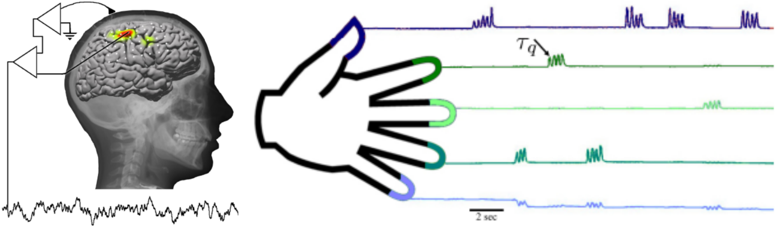






Outline

Motivational example: Brain Computer Interface (BCI)



BCI Competition IV, Dataset 4

- Data: Recordings of ECoG brain signals and of simultaneous finger flexion of a subject (using a glove)
- Objective: predict movement (angle) of the 5 fingers of the subject from its recorded ECoG
- Best performances (at the time!) were obtained using a linear model [Tangermann et al., 2012, Flamary and Rakotomamonjy, 2012]

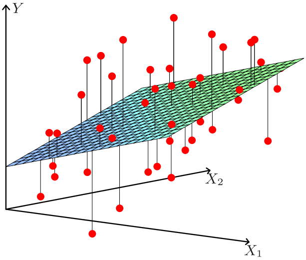
Linear regression

Linear regression model

Find $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d$ such that

$$y_i = \sum_{j=1}^{d} w_j x_{i,j} + \sigma \varepsilon_i, \quad i = 1, \ldots, n$$

$y_i \in \mathbb{R}$
$\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,d})$ fixed, $d < n$
$\varepsilon_i \in \mathbb{R}$, $\mathbb{E}[\varepsilon_i] = 0$, $\mathbb{E}[\varepsilon_i^2] = 1$



Source: [Hastie et al., 2008]

Predictions

Once we estimate the linear coefficient vector, the predictions for a new observation $\mathbf{x}_{new}$ is given by:

$$\hat{y} = \sum_{j=1}^{d} \hat{w}_j x_{new,j} = \mathbf{x}_{new}^\top \hat{\mathbf{w}}$$

Linear regression

Linear regression model

Find $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$ such that

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \sigma\varepsilon$$

$\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$
$\mathbf{X} = (\mathbf{x}^1, \dots, \mathbf{x}^d)$ fixed, $d < n$
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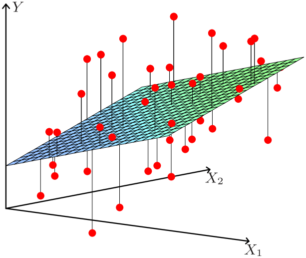

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Notations

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 & 1 \\ \mathbf{x}_2 & 1 \\ \vdots & \vdots \\ \mathbf{x}_i & 1 \\ \vdots & \vdots \\ \mathbf{x}_n & 1 \end{bmatrix} = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,j} & \dots & x_{1,d} & 1 \\ x_{2,1} & x_{2,2} & \dots & x_{2,j} & \dots & x_{2,d} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i,1} & x_{i,2} & \dots & x_{i,j} & \dots & x_{i,d} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,j} & \dots & x_{n,d} & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_n \end{bmatrix}$$

- $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,j}, \dots x_{i,d})^\top$ denotes the features for sample i
- $\mathbf{x}^j = (x_{1,j}, x_{2,j}, \dots, x_{i,j}, \dots, x_{n,j})^\top$ denotes variable j

Least Squares solution

Optimization problem

We want to solve

$$\min_{\mathbf{w}} \quad J(\mathbf{w}) \qquad \text{with} \qquad J(\mathbf{w}) = \frac{1}{2}||\mathbf{y} - \mathbf{X}\mathbf{w}||^2$$

where $J(\mathbf{w})$ is a convex function.

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$\Rightarrow$ Find the parameter $\mathbf{w}$ that leads to a null gradient:

$$\nabla J(\widehat{\mathbf{w}}) = 0 \quad \Leftrightarrow \quad -\mathbf{X}^\top \mathbf{y} + \mathbf{X}^\top \mathbf{X}\widehat{\mathbf{w}} = \mathbf{0}$$

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The solution for Least Squares is the vector $\widehat{\mathbf{w}}^{ls}$ defined as

$$\widehat{\mathbf{w}}^{ls} = \left(\mathbf{X}^\top \mathbf{X}\right)^{-1} \mathbf{X}^\top \mathbf{y}$$

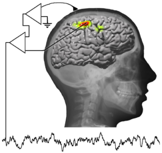
Assumptions

$\mathbf{X}$ is a matrix of rank d (or $d+1$ if bias included) which means that $\mathbf{X}^\top\mathbf{X}$ is invertible.

Sparse linear regression

Issue

What if only a small number of variables are relevant?



Problems

$$\hat{y} = f(\mathbf{x}) = \sum_{j \in J} w_j \mathbf{x}_j$$

- Find a set J of relevant variables
- Estimate the corresponding $\mathbf{w}_J = (w_j)_{j \in J}$

Sparse linear regression

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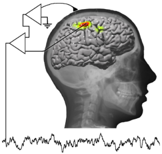


Problems

$$\hat{y} = f(\mathbf{x}) = \sum_{j \in J} w_j \mathbf{x}_j = \sum_{j=1}^{d} w_j \mathbf{x}_j$$

- Find a set J of relevant variables
- Estimate the corresponding $\mathbf{w}_J = (w_j)_{j \in J}$
- For the others: $w_j = 0 \quad \forall j \notin J$

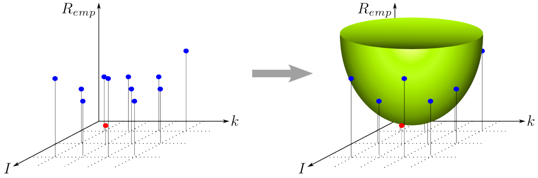
Sparse linear regression

What we would like to do:

$$\begin{cases} \min\limits_{\mathbf{w}\in\mathbb{R}^d} & \dfrac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i^\top \mathbf{w})^2 \\ \text{s.t.} & \#\{w_j \neq 0\} \leq k \end{cases}$$

Issues

- NP-hard problem
- difficult[1] for $d > 40$



[1] One way to avoid the computational burden is to use greedy algorithms

Sparse linear regression

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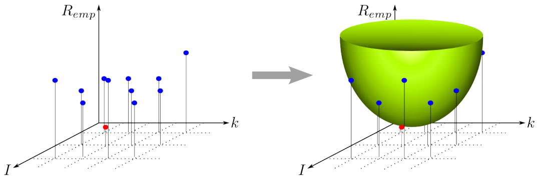

[1] One way to avoid the computational burden is to use greedy algorithms

Sparse linear regression

First approaches:

Best subset: Leaps and bounds (Furnival and Wilson, 1974), branch and bound
Statistical tests:

- Statistical tests for $\hat{w}_j = 0$ (Z-score)
- Statistical tests for J vs J' (F-tests)

Updating set I by adding/removing a variable:

- Forward/backward/stagewise selection [Efroymson, 1960]

Sparse linear regression

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Seminal works on ℓ_1 penalty

- Linear inversion for seismic data [Santosa and Symes, 1986]
- Soft-thresholding [Donoho, 1995]
- Least Absolute Shrinkage and Selection Operator (Lasso) [Tibshirani, 1996]

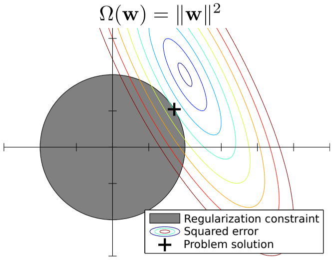
Regularization with the ℓ_2 penalty

Before the ℓ_1 penalty, interesting works had been obtained with the ℓ_2 penalty [Hoerl and Kennard, 1970]

Optimization problems

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i^\top \mathbf{w})^2 \quad s.t. \sum_{j=1}^{d} w_j^2 \leq t$$

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_2^2 \right\}$$



Source: R. Flamary

- Reduce variance by adding bias

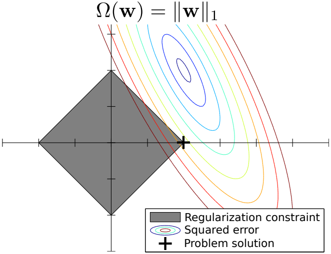
Regularization with the ℓ_1 penalty

Optimization problems

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{x}_i^\top \mathbf{w})^2 \quad s.t. \ \sum_{j=1}^{d}|w_j| \le t$$

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_1 \right\}$$

- Convex relaxation of the ℓ_0 norm
- Simultaneous selection of variables and estimation
- The ℓ_1 norm promotes sparsity



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Diabetes example (sklearn)

Data

- $n = 442$ diabetes patients
- target = quantitative measure of disease progression one year after baseline
- $d = 10$ input variables: age, sex, body mass index, average blood pressure, and six blood serum measurements

```
from sklearn.datasets import load_diabetes
data = load_diabetes()
X = data.data
y = data.target
features = data.feature_names
```

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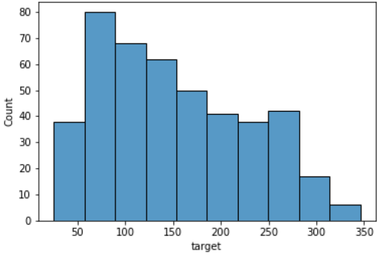
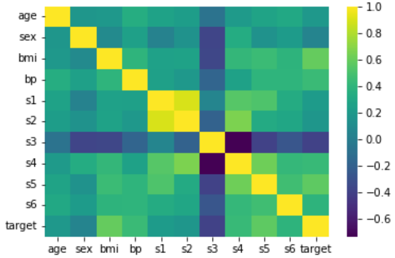
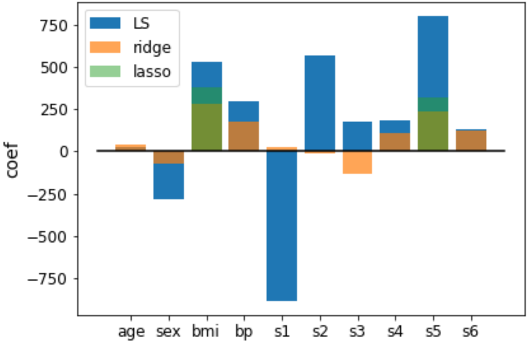




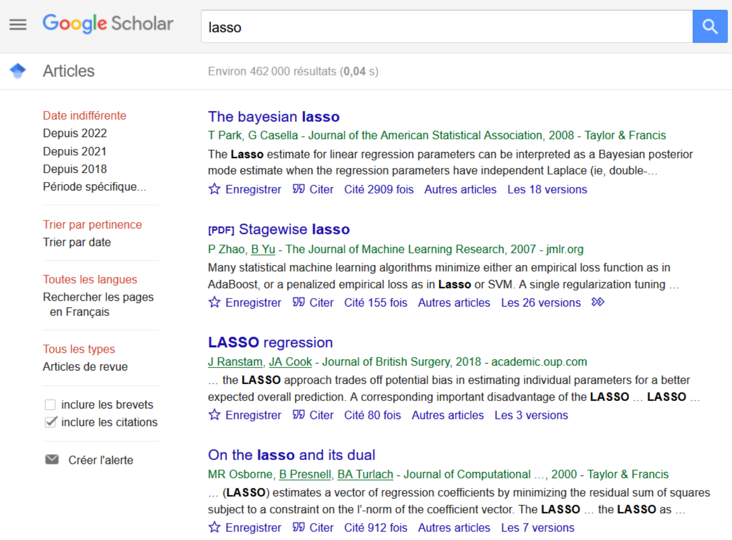
Diabetes example (sklearn)

Estimation of the disease progression

```
from sklearn.linear_model import Lasso
lasso = Lasso(alpha=1)  # default value
lasso.fit(Xtrain, ytrain)
ypred_lasso = lasso.predict(Xtest)
```



Properties of the Lasso



Properties of the Lasso

Special case: $\mathbf{X}$ orthogonal

When $\mathbf{X}$ is orthogonal, we have: $\mathbf{X}^\top \mathbf{X} = I_d$, that is $\mathbf{x}_j^\top \mathbf{x}_j = 1$ and $\mathbf{x}_j^\top \mathbf{x}_l = 0$ for $l \neq j$

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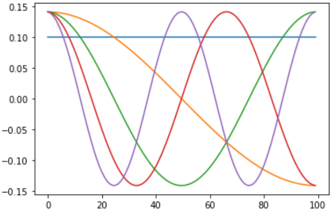
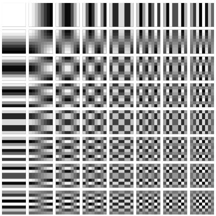
Example: Basis of discrete cosine

$$\cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)\left(k+\frac{1}{2}\right)\right]$$





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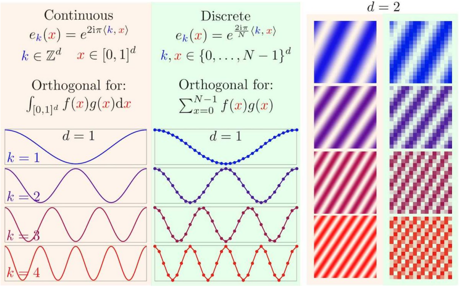
Example: Basis of discrete Fourier



Source: G. Peyré

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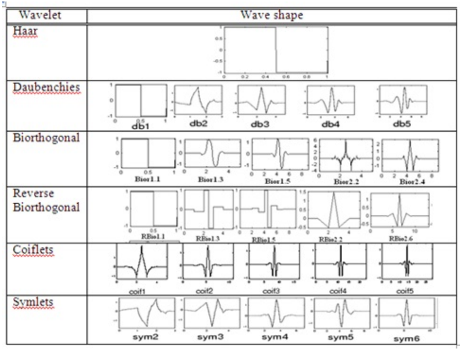
Example: basis of wavelets



Source: [Tarique, 2016]

Properties of the Lasso

Special case: $\mathbf{X}$ orthogonal

- Least squares solution:

$$\hat{\mathbf{w}}^{ls} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} = \mathbf{X}^\top \mathbf{y} \qquad \hat{w}_j^{ls} = \mathbf{x}_j^\top \mathbf{y} \qquad \Rightarrow \qquad \hat{y}^{ls} = \mathbf{x}_{new} \mathbf{X}^\top \mathbf{y}$$

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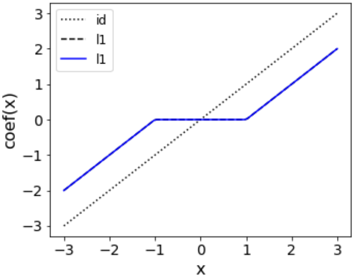
- Lasso solution:

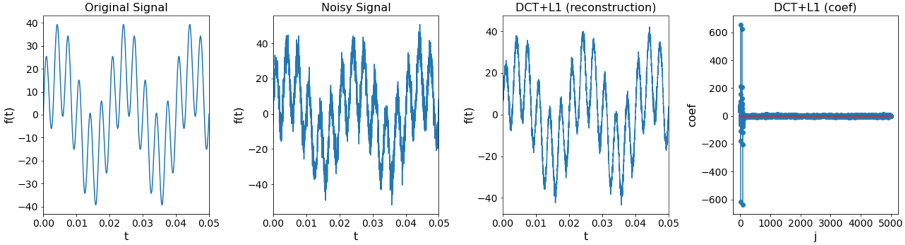
$$\begin{aligned} \hat{w}_j^{lasso} &= (\hat{w}_j^{ls} - \lambda \mathrm{sgn}(\hat{w}_j^{ls}))\mathbb{1}_{\{|\hat{w}_j^{ls}|>\lambda\}} \\ &= \mathrm{sgn}(\hat{w}_j^{ls}) \max\left(|\hat{w}_j^{ls}| - \lambda, 0\right) \end{aligned}$$

$$\hat{y}^{lasso} = \mathbf{x}_{new}^\top \hat{\mathbf{w}}^{lasso}$$



Example with DCT

- data = periodic signal with 2 frequencies (e.g. temperature, tides)
- $n = 5000$ measurements
- noise can come from measurements or external conditions



Solving the Lasso ($\mathbf{X}$ general)

Solving a **differentiable** and **convex** optimization problem is usually performed in 2 steps:

- derive the function to optimize (e.g. in Lagrangian form)
- finding its root by closed form or iteratively e.g. with gradient descent

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Convexity of the Lasso

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ J_{lasso} = \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda\|\mathbf{w}\|_1 \right\}$$

- Sum of 2 convex functions = convex function
- $\mathbf{w} \in \mathbb{R}^d$: convex domain

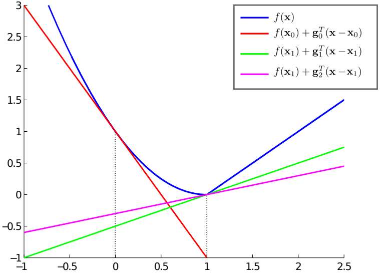
$\Rightarrow$ The problem is therefore convex (but not strictly convex)

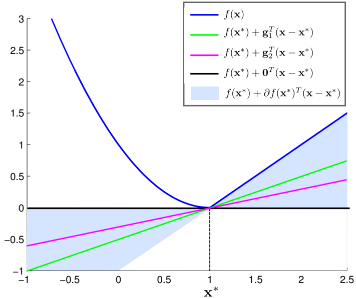
$\Rightarrow$ Any local minimum is also a global minimum

Nondifferentiability of the Lasso

- The absolute value is nondifferentiable in 0

Subgradients and subdifferential



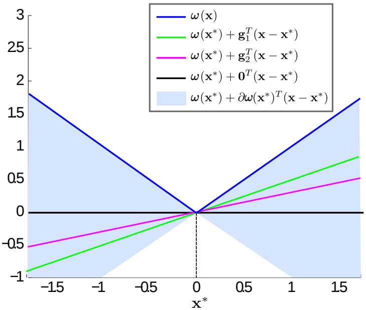


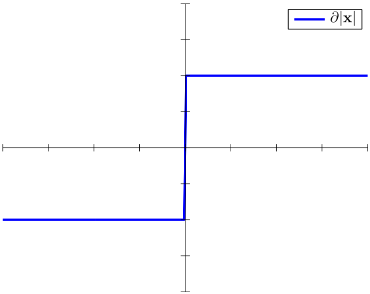
- The notion of gradient can be extended for nondifferentiable functions
- For a convex function $f(\mathbf{x})$, $\mathbf{g}$ is a subgradient of f in $\mathbf{x}_0$ if

$$f(\mathbf{x}) \geq f(\mathbf{x}_0) + \mathbf{g}^\top(\mathbf{x} - \mathbf{x}_0)$$

- The set of all subgradients at $\mathbf{x}_0$ is the subdifferential $\partial f(\mathbf{x}_0)$
- $\mathbf{x}_0$ is a minimum of the convex function f if $\mathbf{0} \in \partial f(\mathbf{x}_0)$

Subdifferential for the ℓ_1 penalty

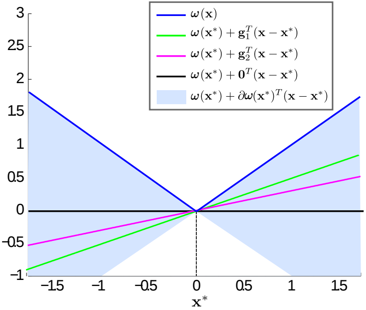


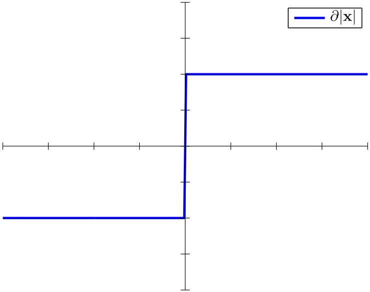


The subdifferential of the absolute value is of the form

$$\partial|x| = \begin{cases} \alpha \in]-1,1[& \text{if } x = 0 \\ sign(x) & \text{if } x \neq 0 \end{cases}$$

Subdifferential for the ℓ_1 penalty





The subdifferential of the ℓ_1 penalty is of the form

$$\partial\|\mathbf{w}\|_1 = (\partial|w_j|)_{j=1}^d = \begin{pmatrix} sign(\mathbf{w}_J) \\ \boldsymbol{\alpha}_{J^c} \end{pmatrix}$$

with J^c the complement pf J *(assuming we know J and the coefficients are reordered)*

Optimality conditions of the Lasso

$\mathbf{w}^\star$ is a solution of the optimization problem if

$$\mathbf{0} \in \partial J_{lasso}(\mathbf{w}^\star) \quad \text{with} \quad J_{lasso}(\mathbf{w}) = \frac{1}{2}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|^2 + \lambda\|\mathbf{w}\|_1$$

This can be reformulated as the following condition

$$-\mathbf{X}^\top(\mathbf{y} - \mathbf{X}\mathbf{w}^\star) + \lambda \mathbf{g} = \mathbf{0} \quad \text{with} \quad \mathbf{g} \in \partial\|\mathbf{w}^\star\|_1$$

Conditions on the components of $\mathbf{w}^\star$

$$\begin{array}{lcr} w_j^\star \neq 0 & \Rightarrow & -\mathbf{x}_j^\top(\mathbf{y} - \mathbf{X}\mathbf{w}^\star) + \lambda \text{sign}(w_j^\star) = 0 \\ w_j^\star = 0 & \Rightarrow & |\mathbf{x}_j^\top(\mathbf{y} - \mathbf{X}\mathbf{w}^\star)| \leq \lambda \end{array}$$

- $\mathbf{x}_j$ is the jth column of $\mathbf{X}$ (feature j).

Lasso with Python

Scikit-Learn

- `sklearn.linear_model.Lasso`: coordinate descent algorithm
- `sklearn.linear_model.SGDRegressor`: stochastic gradient descent
- `sklearn.linear_model.Lars`: least angle regression

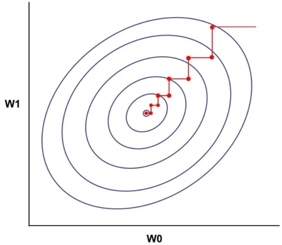
Other Python toolboxes

- `pylops`: ISTA, FISTA and others
- `spams`, `cyanure`: stochastic gradient descent
- `celer.Lasso`, `celer.celer_path`: dual extrapolation

Solvers for Lasso

Coordinate descent (CD)

- Optimize each component of **w** independently until convergence
- Very fast for sparse solutions




```
from sklearn.linear_model import Lasso
lasso = Lasso(alpha=1)  # default value
lasso.fit(Xtrain, ytrain)
ypred_lasso = lasso.predict(Xtest)
```

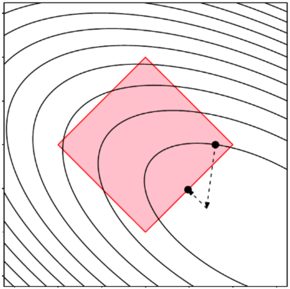
Solvers for Lasso

Proximal gradient descent (PGD)

- Each iteration is a simple soft thresholding of the parameter

$$\mathbf{w}^{(l+1)} = S_\lambda(\mathbf{w}^{(l)} - \gamma \nabla L(\mathbf{y}, \mathbf{X}\mathbf{w}^{(l)}))$$

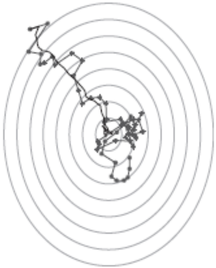
  where $S_\lambda(x) = (x - \lambda)_+$ is the soft-thresholding operator
- (F)ISTA: (Fast) Iterative Soft-Thresholding Algorithm [Daubechies et al., 2010, Beck and Teboulle, 2009]
- Can be coupled with active sets to speedup sparse solutions



Solvers for Lasso

Stochastic gradient descent (SGD)

- Based on proximal algorithms
- Compute the gradient for one sample and optimize for the whole dataset
- Very efficient



```
from sklearn.linear_model import SGDRegressor
lassoSGD = SGDRegressor(penalty='l1', alpha=1)
lassoSGD.fit(Xtrain, ytrain)
ypred_lassoSGD = lassoSGD.predict(Xtest)
```

Regularization path

Aim: find the solution to Lasso for all λ

- Compressed sensing: Basis pursuit denoising [Chen and Donoho, 1994]
- Statistics: Least Angle Regression (LAR) [Efron et al., 2004]

Recall the optimality condition for nonzero coefficients:

$$-\mathbf{X}_J^\top(\mathbf{y} - \mathbf{X}_J\mathbf{w}_J^\star) + \lambda \mathrm{sign}(w_J^\star) = 0$$

Then, assuming we know J and $sign(\mathbf{w}_J)$ (and we can easily compute the inverse matrix):

$$\mathbf{w}_J^\star = (\mathbf{X}_J^\top \mathbf{X}_J)^{-1}\left(\mathbf{X}_J^\top \mathbf{y} - \lambda \mathrm{sign}(\mathbf{w}_J^\star)\right)$$

$\mathbf{w}^*$ is actually linear by parts with respect to λ: we only need to compute it for transition points λ

Regularization path

Aim: find the solution to Lasso for all λ

- Compressed sensing: Basis pursuit denoising [Chen and Donoho, 1994]
- Statistics: Least Angle Regression (LAR) [Efron et al., 2004]

Algorithm

Start: $\mathbf{w}^{(0)} = \mathbf{0}$, $J^{(0)} = \emptyset$, $\lambda^{(0)} = \max_j |\mathbf{x}_j^\top \mathbf{y}|$

Repeat

1. Find vector $\mathbf{x}_j$ most correlated with residual

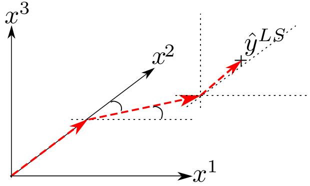
$$\arg\max |\mathbf{x}_j^\top (\mathbf{y} - X_{J^{(l)}} \mathbf{w}^{(l)}_{J^{(l)}})|$$

2. Add it to the set of relevant features

$$J^{(l+1)} \leftarrow J^{(l)} \cup \{j\}$$

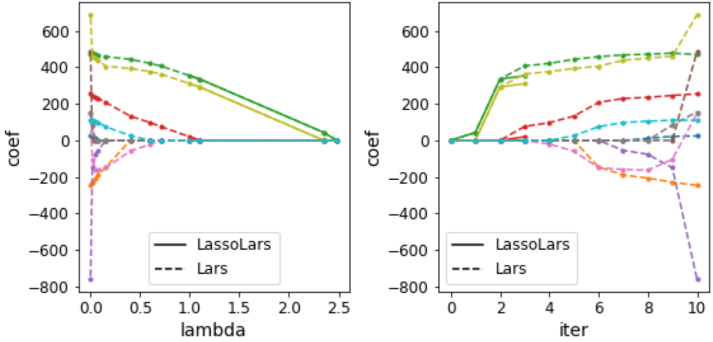
3. Update the coefficients $\mathbf{w}^{(l+1)}_{J^{(l+1)}}$ and $\lambda^{(l+1)}$

until stopping rule.



Diabetes example (sklearn)

```
from sklearn.linear_model import LassoLars, Lars
# Different variants of Lasso regularization path
lasso_lars = LassoLars()  # for full path: set alpha=0
lasso_lars.fit(Xtrain, ytrain)
lars = Lars()
lars.fit(Xtrain, ytrain)
```



Choosing λ

λ tunes the sparsity level:

- $\lambda = 0 \quad \Rightarrow \quad \hat{\mathbf{w}}^{lasso} = \hat{\mathbf{w}}^{ls}$ (all variables are selected)
- $\lambda \to \infty \quad \Rightarrow \quad \hat{\mathbf{w}}^{lasso} = 0$ (no variable is selected)

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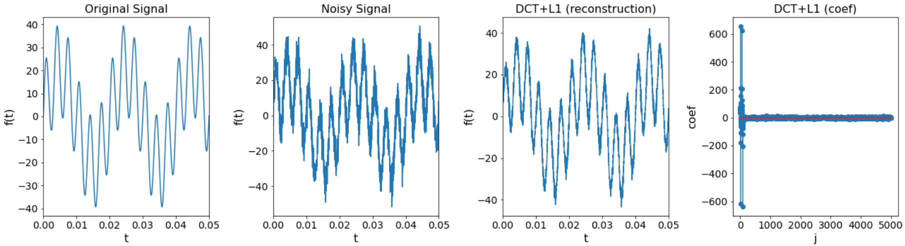

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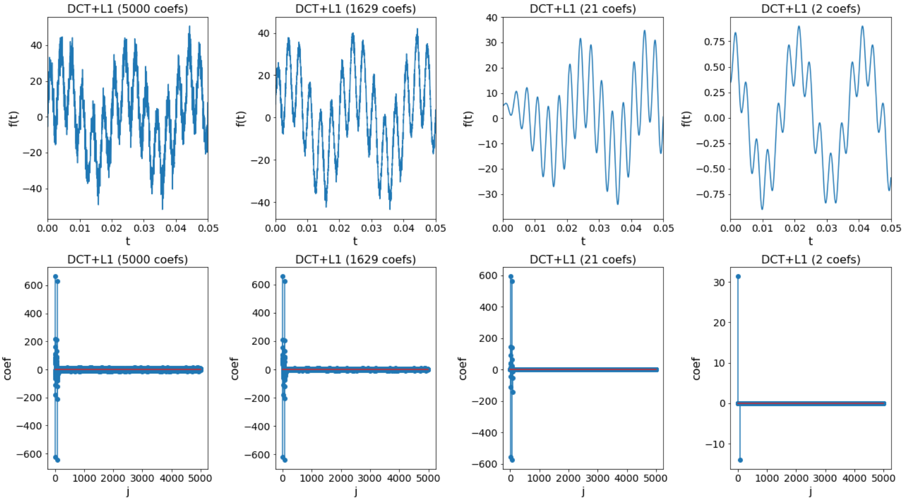

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Measuring the MSE on a different subset

- **Validation**: estimate $\hat{\mathbf{w}}_\lambda$ on train set I_t, find λ minimizing MSE on validation set I_v

$$\lambda^* = \arg\min_{\lambda \geq 0} \frac{1}{n_v} \sum_{i \in I_v} (y_i - \mathbf{x}_i^\top \hat{\mathbf{w}}_\lambda)^2$$

- **Cross-validation**: repeat on K different train/valid partitions

$$\lambda^* = \arg\min_{\lambda \geq 0} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n_v} \sum_{i \in I_v^{(k)}} (y_i - \mathbf{x}_i^\top \hat{\mathbf{w}}_\lambda)^2$$

Choosing λ

λ tunes the sparsity level:

- $\lambda = 0 \quad \Rightarrow \quad \hat{\mathbf{w}}^{lasso} = \hat{\mathbf{w}}^{ls}$ (all variables are selected)
- $\lambda \to \infty \quad \Rightarrow \quad \hat{\mathbf{w}}^{lasso} = 0$ (no variable is selected)

Information Criteria

Use the same set for both $\hat{\mathbf{w}}_\lambda$ and λ

- **Mallow's C_p/Akaike Information Criterion (AIC)**

$$\lambda^* = \arg\min_{\lambda \geq 0} \frac{1}{n_t} \sum_{i \in I_t} (y_i - \mathbf{x}_i^\top \hat{\mathbf{w}}_\lambda)^2 + 2k_\lambda \hat{\sigma}^2$$

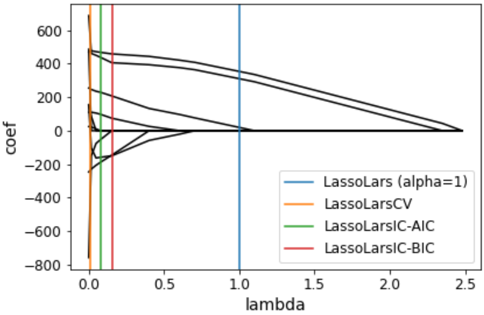
- **Bayes Information Criterion (BIC)**

$$\lambda^* = \arg\min_{\lambda \geq 0} \frac{1}{n_t} \sum_{i \in I_t} (y_i - \mathbf{x}_i^\top \hat{\mathbf{w}}_\lambda)^2 + \log(n_t) k_\lambda \hat{\sigma}^2$$

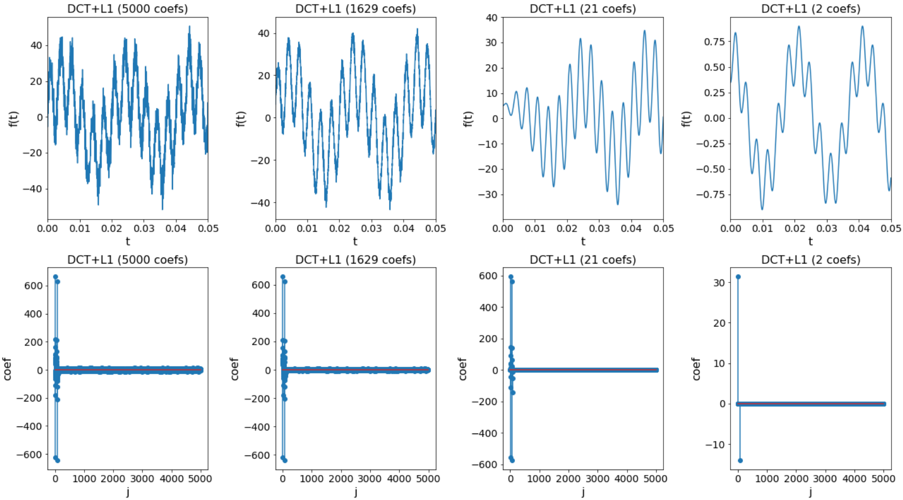
Note: true here because we consider Gaussian errors

Diabetes example (sklearn)

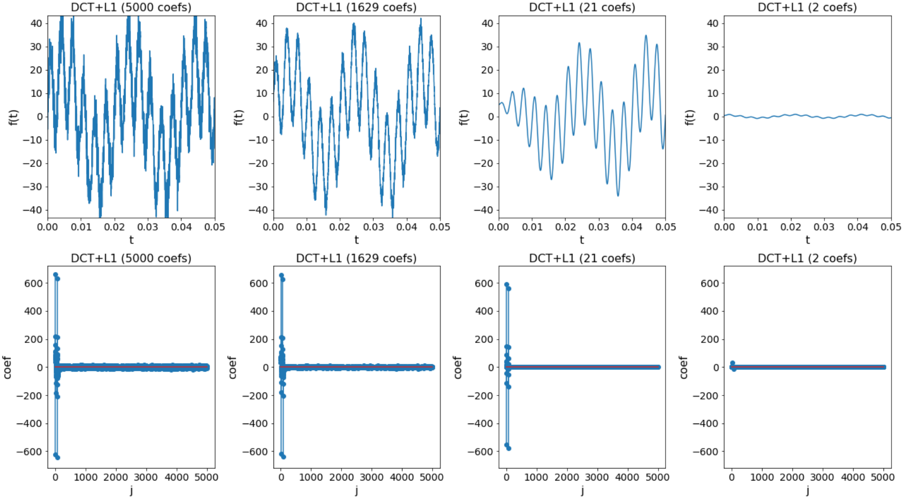
```
from sklearn.linear_model import LassoLars, LassoLarsCV, LassoLarsIC
lasso_lars = LassoLars()
lasso_lars.fit(Xtrain, ytrain)
lasso_larsCV = LassoLarsCV()  # with cross-validation
lasso_larsCV.fit(Xtrain, ytrain)
lasso_larsAIC = LassoLarsIC()  # with AIC or BIC
lasso_larsAIC.fit(Xtrain, ytrain)
lasso_larsBIC = LassoLarsIC(criterion='bic')
lasso_larsBIC.fit(Xtrain, ytrain)
```



Choosing λ



Choosing λ



Issue: Lasso's bias increases with λ

Sparse linear classification

The ℓ_1 penalty can also be applied to classification with other loss functions

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} L(y_i, \mathbf{x}_i\mathbf{w}) + \lambda\|\mathbf{w}\|_1$$

Binary classification

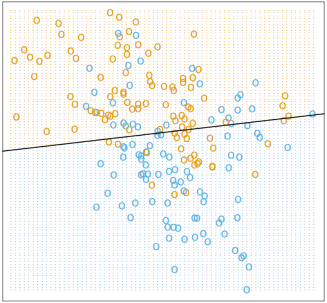
Target takes values: $y_i \in \{-1, 1\}$

- Logistic loss

$$L(y, \mathbf{xw}) = \log(1 + \exp(-y\mathbf{xw}))$$

- Squared hinge loss (SVM type)

$$L(y, \mathbf{xw}) = \max(0, 1 - y\mathbf{xw})^2$$



Source: [Hastie et al., 2008]

Sparse linear classification

The ℓ_1 penalty can also be applied to classification with other loss functions

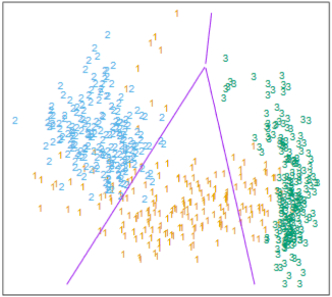
$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} L(y_i, \mathbf{x}_i\mathbf{w}) + \lambda\|\mathbf{w}\|_1$$

Multiclass classification

Target takes values: $y_i \in \{1, \dots, K\}$,
$K =$ no. classes
$\mathbf{W} \in \mathbb{R}^{d\times K}$

- Multinomial logistic regression

$$L(y, \mathbf{x}\mathbf{W}) = -\frac{1}{n}\sum_{i=1}^{n} \log \sum_{k=1}^{K} e^{\mathbf{x}_i^\top(\mathbf{w}^k - \mathbf{w}^{y_i})}$$



Source: [Hastie et al., 2008]

Breast cancer example (sklearn)

Data

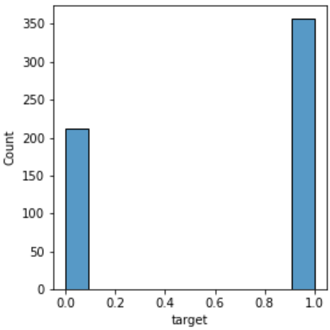
- $n = 569$ patients
- target = detect if mass is malign or benign
- $d = 30$ input variables: mean, std, and "worst" or largest of features (radius, area, texture, perimeter, ...) extracted from images

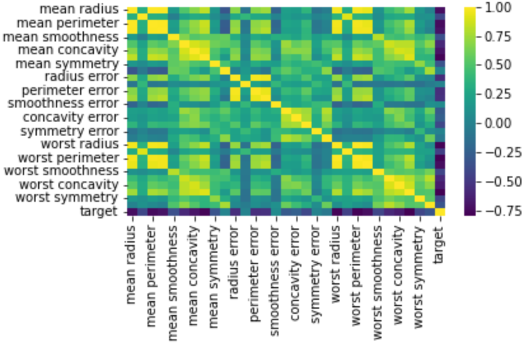
```
from sklearn.datasets import load_breast_cancer
data = load_breast_cancer()
X = data.data
y = data.target
features = data.feature_names
```

Breast cancer example (sklearn)

Data

- $n = 569$ patients
- target = detect if mass is malign or benign
- $d = 30$ input variables: mean, std, and "worst" or largest of features (radius, area, texture, perimeter, ...) extracted from images

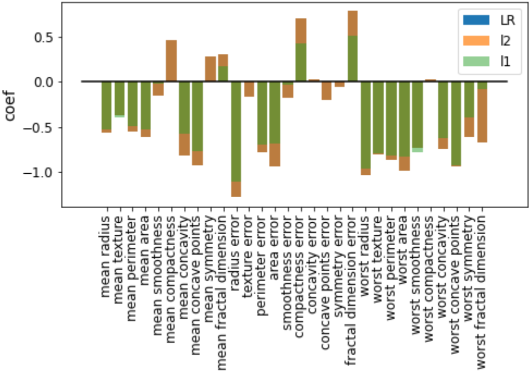




Breast cancer example (sklearn)

Estimation of malignity the mass

```
from sklearn.linear_model import LogisticRegression
l1 = LogisticRegression(penalty='l1', solver='saga', C=1)
l1.fit(Xtrain, ytrain)
ypred_l1 = l1.predict(Xtest)
```



L1 classification with Python

Scikit-Learn

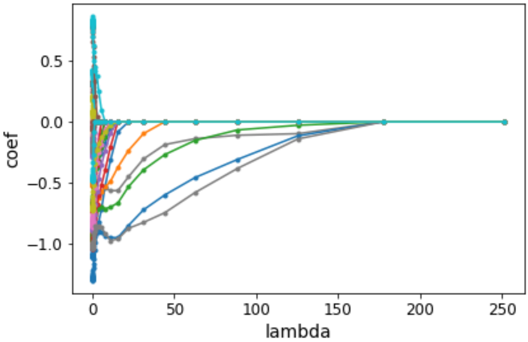
- `sklearn.linear_model.LogisticRegression`: SAGA/liblinear solver
- `sklearn.linear_model.SGDClassifier`: stochastic gradient descent
- `sklearn.svm.LinearSVC(penalty='l1', dual=False)`:
- ✗ regularization path algorithms do not work well here (not piecewise linear): need to compute on a grid

Other Python toolboxes

- `cyanure`: stochastic gradient descent
- `celer.LogisticRegression`, `celer.celer_path`: dual extrapolation

Regularization path

```
from sklearn.linear_model import LogisticRegression
alpha_grid = np.logspace(-5, 2.4, num=50)  # Define a grid
l1_grid = LogisticRegression(penalty='l1', solver='saga', C=1)
coefs_grid = np.zeros((len(l1.coef_[0]), len(alpha_grid)))
for i in range(len(alpha_grid)):
    l1_grid.set_params(C=1/alpha_grid[i])
    l1_grid.fit(Xtrain, ytrain)
    coefs_grid[:, i] = l1_grid.coef_[0]
```



Outline

Sparse penalties

Issue

Lasso is biased, especially for high values of λ (thus very sparse models)

- Can we find better penalties?

Sparse optimization problem

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} h(|w_j|) \right\}$$

- $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is monotously increasing
- $h(|w_j|)$ is nondifferentiable in 0 and assures sparsity
- $h(\cdot)$ does not need to be convex

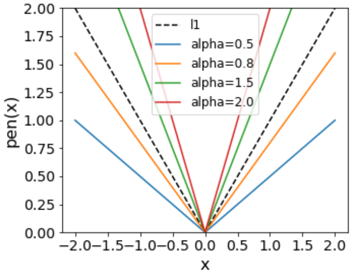
Adaptive Lasso

Convex for fixed weights [Zou, 2006]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda\sum_{j=1}^{d}\alpha_j|w_j|$$

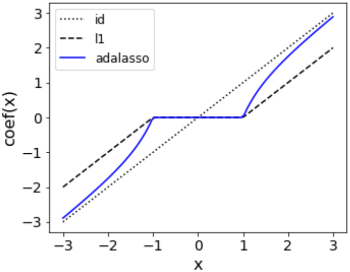
Penalty shape

$$h_j(x) = \alpha_j x \quad \text{with e.g. } \alpha_j = |\hat{w}_j^{ls}|^{-a}$$



Thresholding

$$\hat{w}_j^{adalasso} = \max\left(\hat{w}_j^{ls} - \frac{\lambda \operatorname{sgn}(\hat{w}_j^{ls})}{|\hat{w}_j^{ls}|^a}, 0\right)$$



Reweighted ℓ_1

Weights are not fixed: nonconvex [Candes et al., 2008]

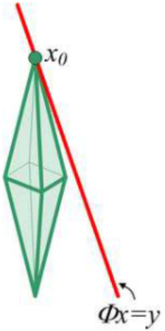
$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} \alpha_j |w_j|$$

Algorithm

Iterate between solving a weighted lasso and updating the weights

$$\begin{aligned}
\hat{\mathbf{w}}^{(l)} &= \min_{\mathbf{w}} \|\hat{\boldsymbol{\alpha}}^{(l)}\mathbf{w}\|_1 \quad \text{s.t.} \quad \mathbf{y}=\mathbf{X}\mathbf{w} \\
\hat{\boldsymbol{\alpha}}_j^{(l+1)} &= \frac{1}{|\hat{w}_j^{(l)}|+\epsilon}, \quad \epsilon>0
\end{aligned}$$

- ϵ ensures stability



Source: [Candes et al., 2008]

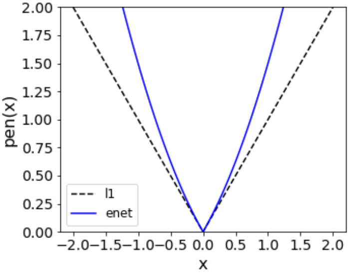
Elastic net ($\ell_1 - \ell_2$)

Strictly convex [Zou and Hastie, 2005]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda_1 \sum_{j=1}^{d} |w_j| + \lambda_2 \sum_{j=1}^{d} w_j^2$$

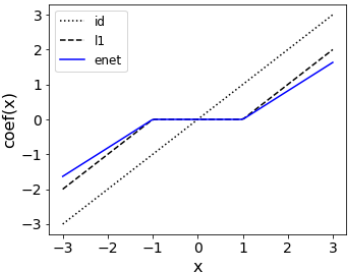
Penalty shape

$$h(x) = x + \rho x^2, \quad \rho = \lambda_2/\lambda_1$$



Thresholding

$$\hat{w}_j^{enet} = \frac{1}{\sqrt{1+\rho\lambda}}\hat{w}_j^{lasso}$$



Adaptive elastic net

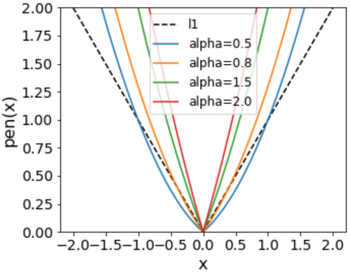
Convex for fixed weights, otherwise nonconvex [Zou and Zhang, 2009]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda_1 \sum_{j=1}^{d} \alpha_j |w_j| + \lambda_2 \sum_{j=1}^{d} \alpha_j w_j^2$$

Penalty shape

$$h_j(x) = \alpha_j x + \rho(\alpha_j x)^2$$

with e.g. $\alpha_j = |\hat{w}_j^{enet}|^{-a}$



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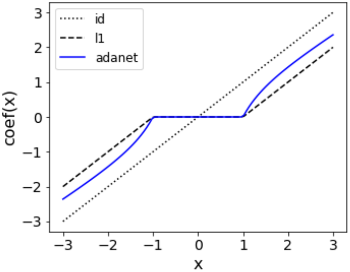


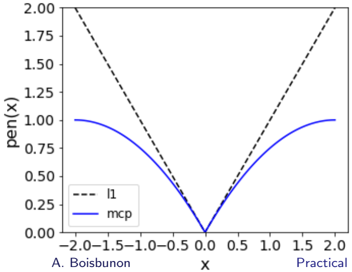
Minimax Concave Penalty (MCP)

Nonconvex penalty [Zhang, 2010], a.k.a semisoft [Gao and Bruce, 1995] or firm shrinkage for $\mathbf{X}$ orthogonal

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} h(|w_j|)$$

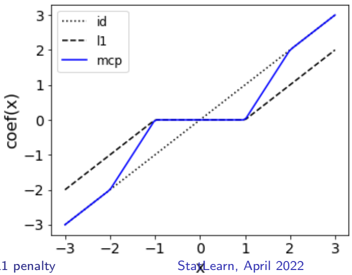
Penalty shape

$$h(x) = \min\left(x - \frac{x^2}{2a\lambda}, \frac{a\lambda}{2}\right),$$
with $a > 1$



Thresholding

$$\hat{w}_j^{mcp} = \frac{a}{a-1}\hat{w}_j^{lasso}\mathbb{1}_{\{|\hat{w}_j^{ls}|\leq a\lambda\}} + \hat{w}_j^{ls}\mathbb{1}_{\{|\hat{w}_j^{ls}|>a\lambda\}}$$



Smoothly Clipped Absolute Deviation (SCAD)

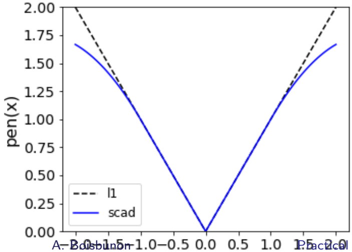
Nonconvex penalty [Fan and Li, 2001]

$$\min_{\mathbf{w} \in \mathbb{R}^d} \frac{1}{n} \|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} h(|w_j|)$$

Penalty shape

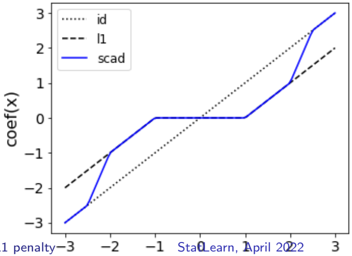
$$\begin{aligned} h(x) &= x \mathbb{1}_{\{x \leq \lambda\}} + \frac{(x-\lambda)^2}{2(a-1)\lambda} \mathbb{1}_{\{\lambda < x \leq a\lambda\}} \\ &+ \frac{(a+1)\lambda}{2} \mathbb{1}_{\{x > a\lambda\}} \end{aligned}$$

with $a > 2$



Thresholding

$$\hat{w}_j^{scad} = \begin{cases} \hat{w}_j^{lasso} & \text{if } |\hat{w}_j^{ls}| \leq 2\lambda \\ \frac{a}{a-2} \hat{w}_j^{lasso} & \text{if } 2\lambda < |\hat{w}_j^{ls}| \leq a\lambda \\ \hat{w}_j^{ls} & \text{if } |\hat{w}_j^{ls}| > a\lambda \end{cases}$$



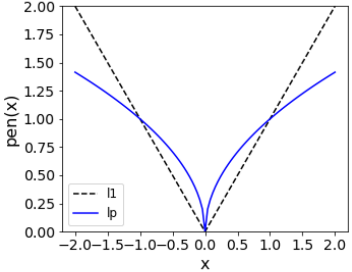
ℓ_p-norm, $0 < p < 1$

[Daubechies et al., 2010]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} |w_j|^p, \qquad 0 < p < 1$$

Penalty shape

$$h(x) = x^p$$



Proximal operator

$$x = sign(x)\theta \quad \text{s.t. } \theta + p\theta^{p-1} = |x|$$

Ingrid Daubechies

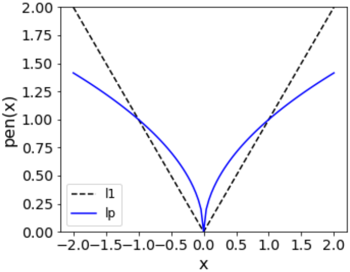
ℓ_p-norm, $0 < p < 1$

[Daubechies et al., 2010]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y} - \mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} |w_j|^p, \qquad 0 < p < 1$$

Penalty shape

$$h(x) = x^p$$



Proximal operator

$$x = sign(x)\theta \quad \text{s.t. } \theta + p\theta^{p-1} = |x|$$

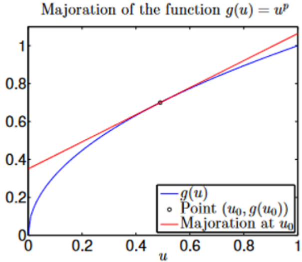


Illustration: [Courty et al., 2014]

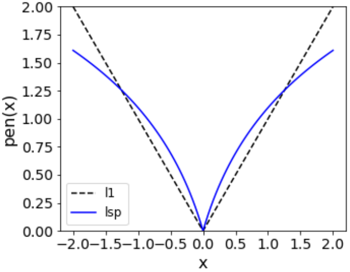
Log-sum penalty (LSP)

[Lobo et al., 2007, Candes et al., 2008]
Proximity operator : [Prater-Bennette et al., 2021]

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\|\mathbf{y}-\mathbf{X}\mathbf{w}\|_2^2 + \lambda \sum_{j=1}^{d} \log\left(1+\frac{|w_j|}{\epsilon}\right)$$

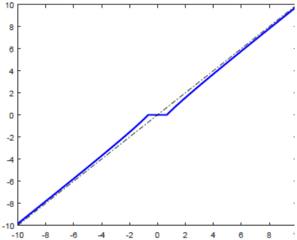
Penalty shape

$h(x) = \log(1 + x/\epsilon)$



Thresholding

$\sqrt{\lambda} \leq \epsilon$



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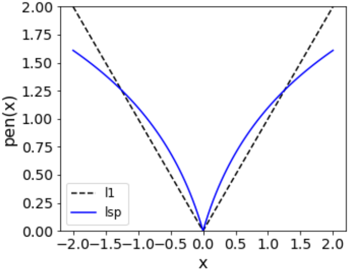
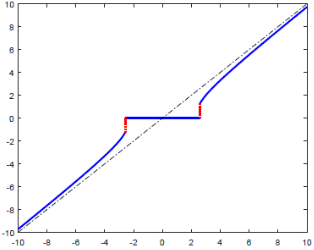
Penalty shape

$h(x) = \log(1 + x/\epsilon)$

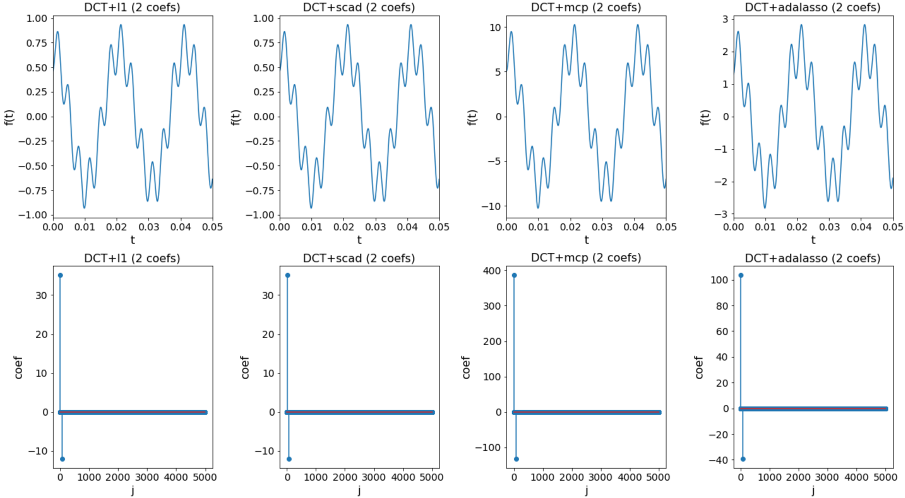


Thresholding

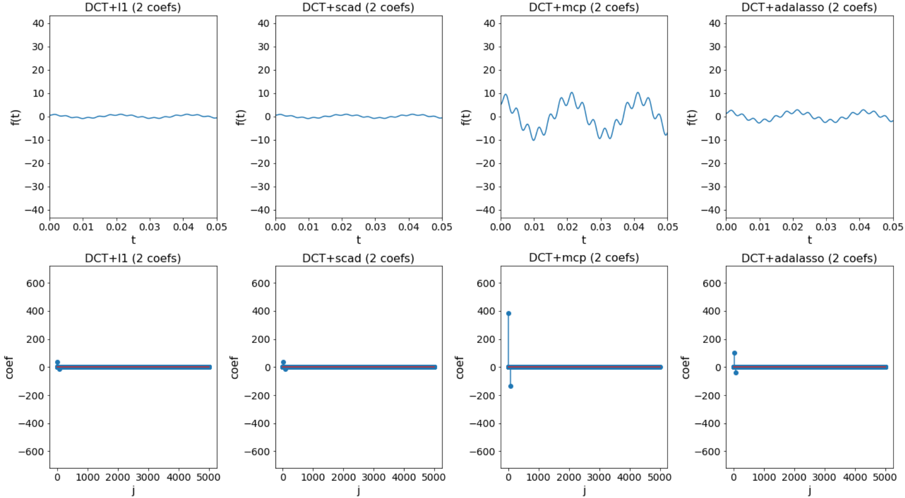
$\sqrt{\lambda} > \epsilon$



Choice of penalty



Choice of penalty



Other interesting sparse penalties

- Group-Lasso [Yuan and Lin, 2006]: promote sparsity on groups of variables

$$\Omega(\mathbf{w}) = \sum_{g=1}^{G} \|\mathbf{w}_{J_g}\|_{K_g} = \sum_{g=1}^{G} (\mathbf{w}_{J_g}^\top K_g \mathbf{w}_{J_g})^{1/2}$$

- Fused lasso [Tibshirani et al., 2005] / Total Variation [Acar and Vogel, 1994]: encourages piecewise constant signals

$$\Omega(\mathbf{w}) = \sum_{j \neq l} |\mathbf{w}_j - \mathbf{w}_l|$$

Optimality conditions

Sparsity for convex penalties

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{\|\mathbf{y} - X\mathbf{w}\|_2^2 + \lambda\Omega(\mathbf{w})\right\}, \qquad \Omega(\mathbf{w}) = \sum_{i=1}^{d} \omega(w_i) \text{ convex}$$

- Non-differentiability in $\mathbf{w} = \mathbf{0}$ with subgradient condition

$$\mathbf{0} \in \partial_{\mathbf{0}}\Omega(\mathbf{w}) = \{\mathbf{g} \in \mathbb{R}^n \setminus \Omega(\mathbf{w}) - \Omega(\mathbf{0}) \geq \mathbf{g}^\top(\mathbf{w} - \mathbf{0})\}$$

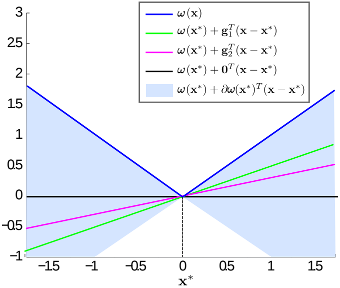
ℓ_1-penalty:

- Subgradient condition

$$\omega(w) = |w| \quad \Rightarrow \quad \partial_0\omega(w) = (-1, 1)$$

- Optimality condition

$$|\left(X^\top(\mathbf{y} - X\mathbf{w})\right)_j| \leq \lambda \quad \forall 1 \leq j \leq p$$



Optimality conditions

Sparsity for non-convex penalties

$$\min_{\mathbf{w}\in\mathbb{R}^d} \left\{ \frac{1}{n}\|\mathbf{y} - X\mathbf{w}\|_2^2 + \lambda\Omega(\mathbf{w}) \right\}, \qquad \Omega(\mathbf{w}) = \|\mathbf{w}\|_1 - h(\mathbf{w})$$

- Non-diff. in $\mathbf{w} = \mathbf{0}$ with Difference of Convex (DC) condition

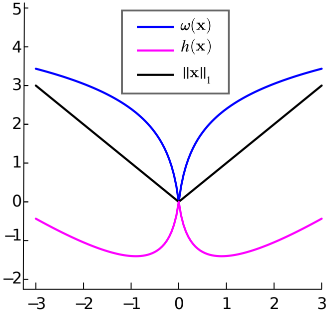
$$\partial_{\mathbf{0}} h(\mathbf{w}) \subset X_J^\top(\mathbf{y} - X\mathbf{w}) + \partial_{\mathbf{0}}\|\mathbf{w}\|_1(\mathbf{w})$$

Log sum penalty (LSP):

$$\omega(w) = \log(1 + |w|/\epsilon)$$

- Optimality condition

$$|\left(X^\top(\mathbf{y} - X\mathbf{w})\right)_j| \leq \lambda/\epsilon \quad \forall 1 \leq j \leq p$$



Sparse penalties with Python

Scikit-Learn

- Elastic net: regression, classification, multitask

Other Python toolboxes

- `cyanure`:
  - Elastic net
  - fused lasso
  - group-lasso
- `celer`:
  - group-lasso (reg)
  - weighted-ℓ_1 (reg/classif)
- `yagml`, `picasso`

Outline

Applications of the ℓ_1 penalty

Many types of problems can be rewritten as a sparse linear problem

Transforming the input variables into new features

- Additive models

$$f(\mathbf{x}) = \sum_{j=1}^{d} w_j \phi_j(x_j)$$

- Modeling interactions between variables

$$f(\mathbf{x}) = \sum_{j'=1}^{d'} w_{j'} \phi_{j'}(\mathbf{x})$$

Applications of the ℓ_1 penalty

Many types of problems can be rewritten as a sparse linear problem

Applying a (nonlinear) transformation to a linear model

- Generalized linear models

$$f(\mathbf{x}) = s\left(\sum_{j=1}^{d} w_j x_j\right)$$

  - Example: activation in a neural network layer

Applications of the ℓ_1 penalty

Many types of problems can be rewritten as a sparse linear problem

Applying sparsity in a different subspace

- Transporting the weights

$$\min_{\mathbf{w}} \|\mathbf{y} - H\mathbf{w}\|_2^2 + \lambda \|\phi^\top \mathbf{w}\|_1$$

Transforming the input variables into new features

Motivation

- linear models are highly interpretable BUT
- Not all data can be estimated/approached with linear regression/classification
- Feature construction is often done "manually" and relies on experts knowledge/ a priori

Random feature generation

- allows to automatically explore possible nonlinearities
- allows to explore features outside prior information

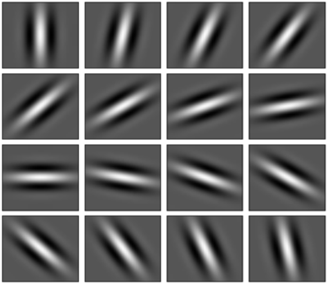
Infinite feature learning

Features generated through Gabor filters, wavelets, Fourier, kernels, etc, with continuous parameter $\theta_j, j = 1, \ldots, d$ [Rakotomamonjy et al., 2013]

$$\mathcal{F} = \{\{\phi_{\theta_j}(\cdot)\}_{j=1}^{d}\}$$

$$f(\mathbf{x}) = \sum_{j=1}^{d} w_j \phi_{\theta_j}(\mathbf{x}) + b_t$$

- Randomly draw d' filters/kernels with different parameters θ_j
- Apply Lasso (or other sparse penalty) to select the most relevant ones
- Repeat step 1 and 2 until convergence



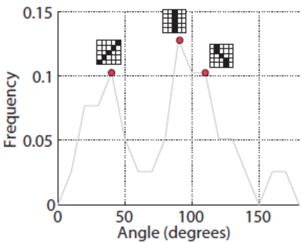
Infinite feature learning

Application: classification of pixels in remote sensing imagery [Tuia et al., 2014]

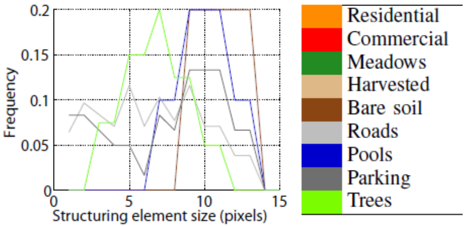
$$\min_{\varphi \in \mathcal{F}} \min_{\mathbf{w}} \frac{1}{n} L(y_i, \mathbf{w}^\top \mathbf{\Phi}_\varphi(\mathbf{x}_i)) + \lambda \|\mathbf{w}\|_1$$

- Classes are types of land cover (one-vs-rest)
- Features generated through Gabor filters
- Different directions and sizes are selected depending on the class

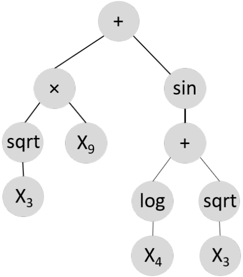
Directions of roads



Size of elements



Symbolic Regression

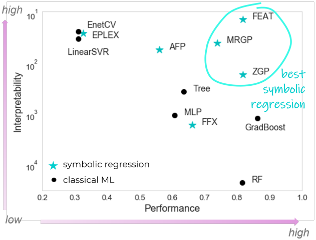


$$\hat{y} = \sqrt{X_3} \cdot X_9 + \sin(\log X_4 + \sqrt{X_3})$$

Search models with analytical form

- Ability to model interactions between variables
- Ability to construct new features
- Good tradeoff between interpretability and flexibility

Symbolic Regression and ℓ_1 penalty



Some SR algorithms randomly build features and then linearly combine them with a sparse penalty to select the most relevant ones

- features = bases in the form $\{op(X_i)\}_{i=1}^p$
  - FFX – Fast Function eXtractor [McConaghy, 2011]
- construct/evolve branches via genetic programming (GP)
  - MRGP – Multiple Regression GP [Arnaldo et al., 2014]
  - FEAT – Feature Engineering Automation Tool [La Cava et al., 2018]
  - ZGP – Zoetrope GP [Boisbunon et al., 2021]

Compressed Sensing with random features

Universal encoding [Candes and Tao, 2006]

Motivation

- Reduce data acquisition, e.g. for Magnetic Resonance Imaging, or for allowing satellite imaging with low transmission rates
- Ensure security in encoder-decoder

Principle

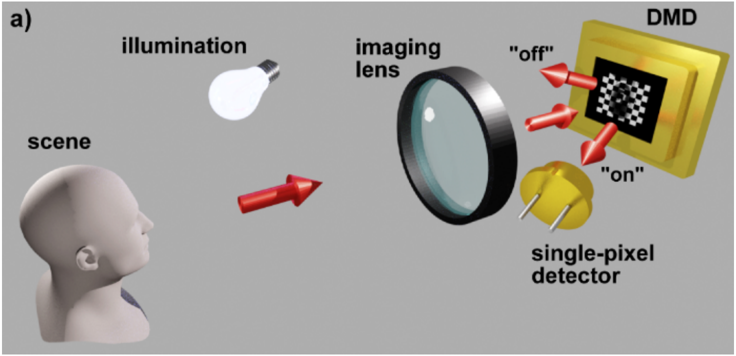
- Generate a collection of random vectors $\mathbf{x}_k$, e.g. random rows of Fourier or random Gaussian vectors
- Share the collection (e.g. send the seed)
- Encoder part: $y_k =< f, \mathbf{x}_k > +$ apply quantization
- Decoder part: Lasso

Compressed Sensing with random features

Single pixel imaging

[Gibson et al., 2020]

- cheaper sensors than traditional sensor arrays (e.g. for infrared)
- ability to detect weak light intensity changes



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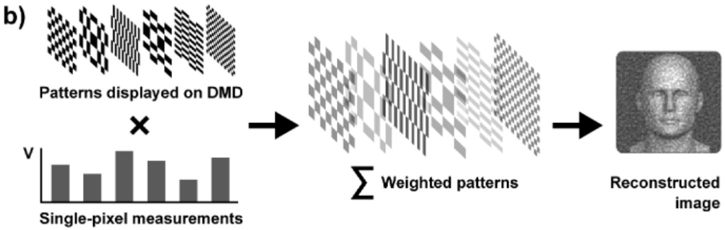


ℓ_1 penalty and Neural Networks

Sparsity in neural networks

- In neural networks, we apply nonlinear transformations to linear models
- The linear layers in a neural network can also be sparsified through ℓ_1
  - limit overfitting
  - concentrate the learning to the most important connexions between neurons

Using ℓ_1 penalty with Keras

```
from tensorflow.keras import layers
from tensorflow.keras import regularizers

tf.keras.regularizers.l1(l1=0.01)
tf.keras.regularizers.l1_l2(l1=0.01, l2=0.01)
```

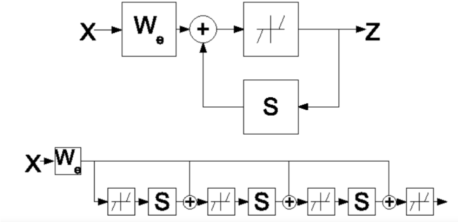
ℓ_1 penalty and Neural Networks

Real-time applications

- Sometimes computing the sparse weights may be too long for real time applications (e.g. in telecom)

Learning ISTA and CD [Gregor and LeCun, 2010]

- Run (F)ISTA/CD or your favorite algorithm on the dataset
- Train a neural network that predicts the results of the (F)ISTA/CD/etc



Signal/image processing

Denoising

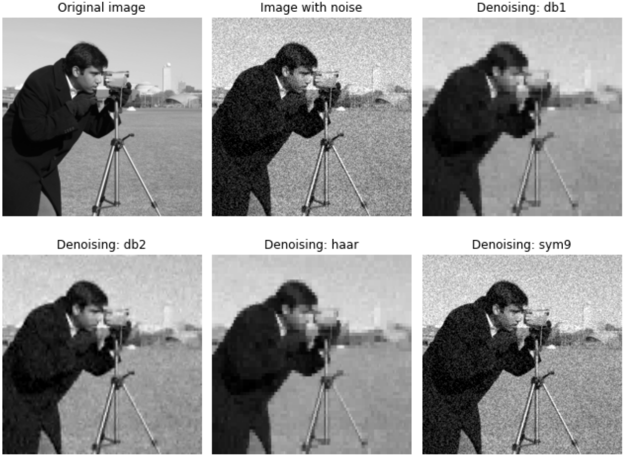
$$\mathbf{y} = \mathbf{x} + \varepsilon$$

- Aim: recover original signal $\mathbf{x} = \mathbf{D}\mathbf{w}$ from noisy observations $\mathbf{y}$
- $\mathbf{D}$ is a (fixed) dictionary
- Regular setting for Lasso

Source: R. Flamary

Denoising with wavelets in Python

```
from skimage.restoration import denoise_wavelet
denoised_img = denoise_wavelet(noisy_img, wavelet='db1',
    mode='soft', method='BayesShrink')
```

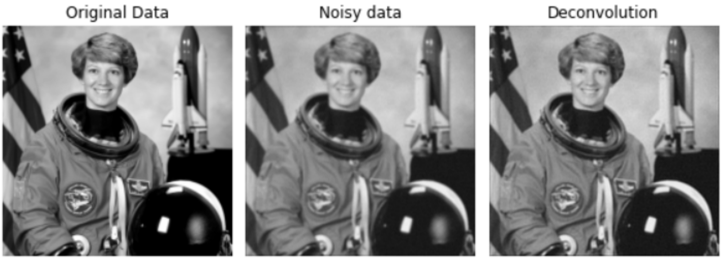


Signal/image processing

Reconstructing a signal

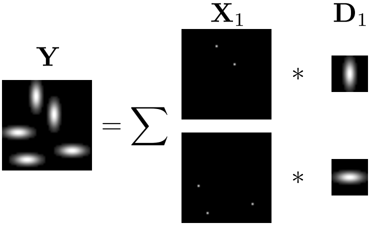
$$\mathbf{y} = \mathbf{H}\mathbf{x} + \varepsilon$$

- Aim: recover original signal $\mathbf{x} = \mathbf{D}\mathbf{w}$ from noisy observations $\mathbf{y}$
- $\mathbf{D}$ is a (fixed) dictionary
- $\mathbf{H}$ is a known linear operator, e.g. convolution or blur operator



Application: object detection

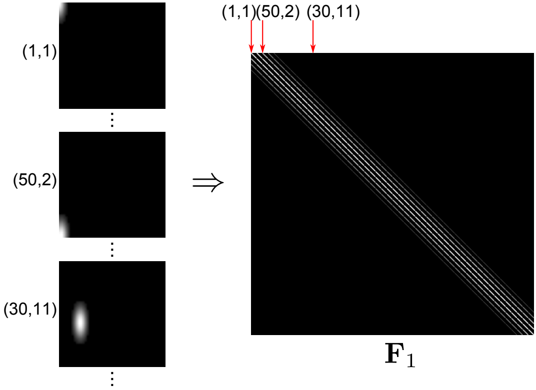
Detection of objects (boats) from satellite images with fixed dictionary [Boisbunon et al., 2014b]



- $\mathbf{Y}$: matrix of size $n \times m$ (image)
- $\mathbf{D}_k$, $k = 1, \ldots, K$: dictionary atoms
- $\mathbf{X}_k$: extremely sparse matrix of size $n \times m$
  - $x_{i,j,k} \neq 0 \Rightarrow$ position (i, j) activated for atom $\mathbf{D}_k$
- Reconstructed image: $\widetilde{\mathbf{Y}} = \sum_{k=1}^{K} \mathbf{X}_k * \mathbf{D}_k$

Application: object detection

Equivalence with linear problem



- Sum of convolutions: $\min_{\mathbf{X}\in\mathbb{R}_+^{n\times m\times K}} \left\{\|\mathbf{Y} - \sum_{k=1}^{K} \mathbf{X}_k * \mathbf{D}_k\|_F^2 + \lambda\Omega(\mathbf{X})\right\}$
- Linear problem: $\min_{\mathbf{x}\geq 0} \left\{\|\mathbf{y} - \mathbf{F}\mathbf{x}\|_2^2 + \lambda\Omega(\mathbf{x})\right\}$

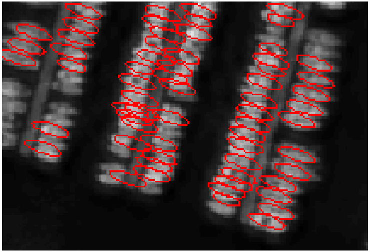
Application: object detection

Algorithm

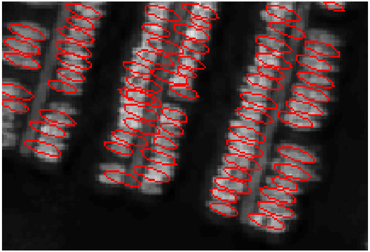
2D Sparse Optimization (2DSO) with active set strategy

- Find the atom most correlated with image $\Rightarrow$ shape + position
- Add the atom to active set
- Solve problem on a small active set[2] (verify optimality conditions) and apply transformation vector $\rightarrow$ matrix

ℓ_1-penalty



Log-sum penalty



[2] [Boisbunon et al., 2014a]

Dictionary learning

Aim: reconstruct $\mathbf{X} = \mathbf{DW}$ with both $\mathbf{D}$ and $\mathbf{W}$ unknown

Optimization problem

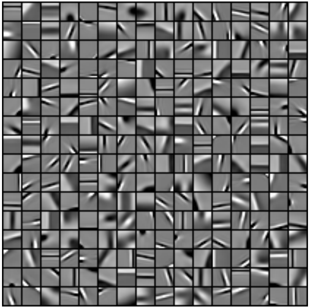
$$\min_{\mathbf{W}\in\mathbb{R}^{d\times m}, \mathbf{D}\in\mathbb{R}^{n\times d}, \|\mathbf{d}\|_j=1} \|\mathbf{Y} - \mathbf{DW}\|_F^2 + \lambda \sum_{j=1}^{d} \|\mathbf{w}_j\|_1$$

Algorithm

Start: $\mathbf{W}^{(0)} = \mathbf{0}$, $\mathbf{D}^{(0)}$

1. Extract patches from image
2. *Repeat*
   - Solve optimization problem for $\mathbf{W}^{(l+1)}$ with $\mathbf{D}^{(l)}$ fixed
   - Solve optimization problem for $\mathbf{D}^{(l+1)}$ with $\mathbf{W}^{(l+1)}$ fixed

until stopping rule.



Source: [Bach et al., 2011]

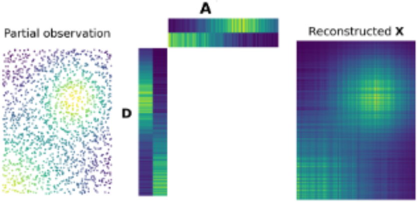
Application: inpainting

Dictionary learning with a mask

[Mairal et al., 2008]

$$\min_{\mathbf{W}\in\mathbb{R}^{d\times m},\mathbf{D}\in\mathbb{R}^{n\times d},\|\mathbf{d}\|_j=1} \|\mathbf{M}\odot(\mathbf{Y}-\mathbf{DW})\|_F^2+\lambda\sum_{j=1}^{d}\|\mathbf{w}_j\|_1$$

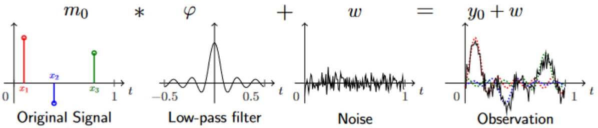
- $\mathbf{M}$ = binary mask of pixels we wish to recover
- $\odot$ is the pointwise multiplication



Beurling Lasso

Input = Blurred observations from measuring devices

- Beurling allows to explore Diracs in continuous space



Reconstruct a discrete measure from noisy samples [Azais et al., 2015]

$$\min_{\mu} \| \int \phi d\mu - \mathbf{y} \|^2 + \lambda \|\mu\|_{TV}$$

$$\min_{\mu} \|\phi\mu - \mathbf{y}\|^2 + \lambda |\mu|(\mathcal{X})$$

Image courtesy of S. Ladjal

Outline

Other interesting works with sparse penalties

Some applications of ℓ_1 I did not mention but are very interesting too:

- Selecting the k best singular value for matrix factorization, e.g. in recommandation systems
- Analysis of spike trains in the brain with Hawkes processes [Reynaud-Bouret et al., 2013]
- Sparse subspace clustering [Elhamifar and Vidal, 2013]
- Multitask learning
- Unbalanced optimal transport [Chapel et al., 2021]

and many more!

Take-home messages

- ℓ_1 penalty is everywhere!
- always try simple/classical approaches first (baseline)
- research is not always about new ideas, it can also be about how to adapt it in a new framework/context

Outline

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Pursuit algorithms

X = overcomplete dictionary of p atoms (wavelets, Gabor, Fourier), $p > n$

Matching pursuit [Mallat and Zhang, 1993]

$$\min_{\mathbf{w}} \|\mathbf{y} - X\mathbf{w}\|_2^2 \qquad s.t. \quad \|\mathbf{w}\|_0 \leq p^*,$$

regularization path where the coefficients are updated with $w_j = \mathbf{x}_j^\top \mathbf{y}$

Basis pursuit

$$\min_{\mathbf{w}} \|\mathbf{w}\|_1 \qquad s.t. \quad \mathbf{y} = X\mathbf{w}$$

Basis pursuit denoising [Chen and Donoho, 1994]

$$\min_{\mathbf{w}} \|\mathbf{w}\|_1 \qquad s.t. \quad \|\mathbf{y} - X\mathbf{w}\|_2^2 \leq t_\lambda$$

with $t_\lambda = \sigma\sqrt{2\log(p)}$ Jump to reg path slide